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# Nonstandard arguments and the stability of generic structures

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## Abstract

Generic 構造の研究に超準的手法を導入する. 本稿では特に, Wagner が行った generic 構造の安定性の強さについての研究 [4] に, 弱冠の新しい結果を付け足す. Wagner は saturated な generic 構造が安定になる為の十分条件 DS と  $\omega$ -安定になる為の十分条件 DW を定義した. 本稿では DS を簡略化し, DS と DW の間の関係を調べる.

## 1 Preliminaries

Let  $L$  be a countable relational language. Let  $\mathbb{K}$  be a nonempty class of finite  $L$ -structures closed under isomorphisms and substructures (we consider the emptyset as an  $L$ -structure). Suppose  $A \leq B$  is a reflexive and transitive relation on elements  $A \subseteq B$  of  $\mathbb{K}$ , which is invariant under isomorphisms. If  $A \leq B$  holds, we say that  $A$  is closed in  $B$ . We also assume that  $(\mathbb{K}, \leq)$  satisfies the following properties:

1.  $\emptyset \leq A$ ,
2.  $A \subseteq B \subseteq C, A \leq C \implies A \leq B$ ,
3.  $A \leq B \implies A \cap C \leq B \cap C$ .

Let  $(\mathbb{K}, \leq)$  be as above. Let  $N$  be an  $L$ -structure whose any finite substructure belongs to  $\mathbb{K}$ . Note that for any  $A \subseteq N$ , there is a unique smallest closed superset of  $A$  in  $N$ . We call this set the closure of  $A$ .

**Definition 1** *Let  $A \subseteq B$ . We say that  $B$  is a minimal extension of  $A$  if the following conditions are satisfied*

- $A \not\leq B$

- $A \leq B'$  for any  $A \subseteq B' \subset B$ .

**Definition 2** Let  $\leq$  be a closed relation on  $\mathbb{K}$ . Then we say that  $(\mathbb{K}, \leq)$  satisfies finite closure axiom if there is no infinite chain  $(A_i)_{i < \omega}$  of elements of  $\mathbb{K}$  such that  $A_{i+1}$  is a minimal extension of  $A_i$  for each  $i < \omega$ .

We assume that  $(\mathbb{K}, \leq)$  satisfies the finite closure axiom in this paper. We say that an  $L$ -structure  $N$  has finite closures if for any finite  $A \subseteq N$ , the closure of  $A$  is also finite. Put  $\bar{\mathbb{K}} = \{N : L\text{-structure} \mid A \in \mathbb{K} \text{ for any } A \subset_{\text{fin}} N\}$ .

**Fact 3** [2] Let  $\leq$  be a closed relation on  $\mathbb{K}$ . Then the following are equivalent:

1.  $\mathbb{K}$  satisfies finite closure axiom.
2. Every member of  $\bar{\mathbb{K}}$  has finite closures.
3. Every  $\omega$ -saturated member of  $\bar{\mathbb{K}}$  has finite closures.
4. Some  $\omega$ -saturated member of  $\bar{\mathbb{K}}$  has finite closures.

**Definition 4** Let  $M$  be an  $L$ -structure. We say that  $M$  is a  $\mathbb{K}$ -generic structure if the following conditions are satisfied:

1.  $M$  is countable.
2.  $\forall A \subset_{\text{fin}} M, A \in \mathbb{K}$  (i.e.  $M \in \bar{\mathbb{K}}$ ).
3.  $A \leq M, A \leq B \in \mathbb{K} \Rightarrow \exists B' \leq M$  such that  $B' \cong_A B$ .

**Fact 5** Suppose that  $(\mathbb{K}, \leq)$  satisfies the finite closure axiom. Then a  $\mathbb{K}$ -generic structure is unique.

**Definition 6** Let  $d$  be a function from  $\{A : A \leq_{\text{fin}} M\}$  to  $\mathbb{R}_{\geq 0}$ . We say  $d$  is a dimension function for  $M$  if for all  $A, B \leq_{\text{fin}} M$ ,

1.  $A \subset B \Rightarrow d(A) \leq d(B)$
2. (Monotonicity)  $d(\overline{A \cup B}) + d(A \cap B) \leq d(A) + d(B)$
3.  $A \cong B \Rightarrow d(A) = d(B)$

For arbitrary  $A \subset_{\text{fin}} M$ , we put  $d(A) = d(\overline{A})$ . We define  $d(A/B)$  the relative dimension of  $A$  over  $B$ . For finite  $A, B$ ,  $d(A/B) = d(AB) - d(B)$ . For finite  $A$ , arbitrary  $B$ ,  $d(A/B) = \inf\{d(A/B_0) : B_0 \subset_{\text{fin}} B\}$ . It is easy to check that these two definitions has the same value in the case  $A$  and  $B$  are finite.

## 2 Nonstandard arguments

Let  $M$  be the  $\mathbb{K}$ -generic structure and  $d$  be a dimension function for  $M$ . We consider  $M$  to be a 3-sorted structure

$$(M \cup P \cup \mathbb{R}; F, \in, d \leq, \dots)$$

where  $P, F, \in$  are as above,  $d$  is the dimension function of  $M$ ,  $\leq$  is the closed relation on  $P \times P$ .

We define the nonstandard model  $M^*$  of  $M$  by a sufficiently saturated extension of this structure

$$(M \cup P \cup \mathbb{R}, F, \in, d \leq, \dots) \prec (M^* \cup P^* \cup \mathbb{R}^*, F^*, \in^*, d^*, \leq^* \dots)$$

**Definition 7** A set  $A \in F^*$  is said to be a hyperfinite set. For  $A \subseteq M$ ,  $A^* \in F^*$  is said to be a hyperfinite extension of  $A$  if

- $M^* \models a \in^* A^*$  for each  $a \in A$ , and
- $M^* \models A^* \subseteq^* A$ .

write  $A \subset_{hf} A^*$ ,  $A^* \supset_{hf} A$

By saturation, a hyperfinite extension of  $A$  always exists.

**Lemma 8** For any subseteq  $A$  of  $M$ , there exists a hyperfinite extension of  $A$ .

*Proof:* It is enough to prove that the following set of formulas is satisfiable:

$$\Gamma(X) = \{a \in^* X \mid a \in A\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$

But for any finite subseteq  $A_0$  of  $A$ ,  $A_0$  realizes the following set of formulas:

$$\{a \in^* X \mid a \in A_0\} \cup \{X \subseteq^* A\} \cup \{X \in F\}.$$

So, by compactness,  $\Gamma(X)$  is satisfiable.

Let  $x, y$  be two nonstandard (or standarad) real numbers. We write  $x \approx y$  if  $|x - y| < 1/n$  for each  $n \in \omega$ .

**Lemma 9** For  $r \in \mathbb{R}$ ,  $\bar{a} \in M$  and  $A \subset M$ , the following are equivalent.

1.  $d(\bar{a}/A) = r$ ;

2.  $d^*(\bar{a}/A^*) \approx r$  for any  $A^* \supset_{hf} A$ ;
3.  $d^*(\bar{a}/A^*) \approx r$ , for some  $A^* \supset_{hf} A$ .

*Proof:* (1  $\rightarrow$  2): By monotonicity of  $d$ , there are  $A_n \subset_{fin} A$  ( $n = 1, 2, \dots$ ) such that  $\forall X \in F$

$$A_n \subset X \subset A \rightarrow r \leq d(\bar{a}/X) \leq r + 1/n.$$

These statements hold also in  $M^*$ . So if  $A^*$  is a hyperfinite extension of  $A$ , then we have

$$r \leq d^*(\bar{a}/A^*) \leq r + 1/n \quad (n = 1, 2, \dots)$$

So we have  $d^*(\bar{a}/A^*) \approx r$ .

(2  $\rightarrow$  3): trivial.

(3  $\rightarrow$  1): We assume 3 and choose a witness  $A^*$ . Then  $(d^*(\bar{a}/A^*) \approx r)$ . Suppose 1 is not the case. Then there is  $s \neq r$  such that  $d(\bar{a}/A) = s$ . By 1  $\Rightarrow$  2, we have  $d^*(\bar{a}/A^*) \approx s$ . A contradiction.

Note that  $M \models \forall A \in P \exists! \bar{A} (A \subseteq \bar{A} \leq M \wedge \forall X A \subseteq X \leq M \rightarrow \bar{A} \subseteq X)$ . This formula holds also in  $M^*$ . For  $X \in P^*$ , we write  $\bar{X}$  as the "closure" of  $X$  in  $M^*$ . In this paper,  $M \models X \in F^* \rightarrow \bar{X} \in F^*$  because  $\mathbb{K}$  satisfies the finite closure condition.

### 3 Main result

**Definition 10** ([4])

1. Let  $A, B \subset_{fin} M$  and  $C \subset M$ . Then we say  $A$  and  $B$  are  $d$ -independent over  $C$  and write  $A \perp_C^d B$  if the following conditions are satisfied:
  - $d(A/BC) = d(A/C)$ , and
  - $\overline{AC} \cap \overline{BC} = \overline{C}$ .
2. For arbitrary  $A, B, C \subset M$ , we say  $A$  and  $B$  are  $d$ -independent over  $C$  if for each  $A_0 \subset_{fin} A, B_0 \subset_{fin} B, A_0 \perp_C^d B_0$

Note that for closed sets  $A, B$ ,  $A$  and  $B$  are  $d$ -independent over  $A \cap B$  if and only if for each  $A_0 \subset_{fin} A, B_0 \subset_{fin} B, d(A_0/B_0(A \cap B)) = d(A_0/A \cap B)$ .

**Definition 11** Let  $A$  and  $B$  be closed subsets of  $M$ . Then we say  $A$  and  $B$  are  $d^*$ -independent over  $A \cap B$  if the following conditions are satisfied: there exist a hyperfinite extension  $A^*$  of  $A$  and a hyperfinite extension  $B^*$  of  $B$  such that

- $A^*$  and  $B^*$  are both closed
- $d(A^*/B^*) = d(A^*/A^* \cap B^*)$

Wagner's definition of DS (a sufficient condition for saturated  $M$  to be stable) is as follows:

For any closed  $A, B$ , if  $\forall n \in \omega, \forall A_0 \subset_{\text{fin}} A, \forall B_0 \subset_{\text{fin}} B, A_0 \subset \exists A' \leq_{\text{fin}} A, B_0 \subset \exists B' \leq_{\text{fin}} B$  such that

$$d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n,$$

then  $A$  and  $B$  are free over  $A \cap B$  and  $AB$  is closed.

On the other hands, Wagner's definition of DW (a sufficient condition for saturated  $M$  to be  $\omega$ -stable) is as follows:

- for any closed  $A, B$ , if  $A \downarrow_{A \cap B}^d B$ , then  $A$  and  $B$  are free over  $A \cap B$  and  $AB$  is closed and
- for any  $\bar{a}$  and  $X$ , there exists finite  $X_0 \subseteq X$  such that  $d(\bar{a}/X_0) = d(\bar{a}/X)$ .

**Theorem 12** For arbitrary closed  $A, B$ , the following are equivalent:

1.  $\forall n \in \omega, \forall A_0 \subset_{\text{fin}} A, \forall B_0 \subset_{\text{fin}} B, A_0 \subset \exists A' \leq_{\text{fin}} A, B_0 \subset \exists B' \leq_{\text{fin}} B$  such that  $d(A') + d(B') \leq d(A'B') + d(A' \cap B') + 1/n$
2.  $A \downarrow_{A \cap B}^{d^*} B$
3.  $A \downarrow_{A \cap B}^d B$

*Proof:* (1  $\rightarrow$  2): Assume 1. Then by saturatedness, There exist a closed hyperfinite extension  $A^*$  of  $A$  and a closed hyperfinite extension  $B^*$  of  $B$  such that for all  $n \in \omega$ ,

$$d^*(A^*) + d^*(B^*) \leq d^*(A^*B^*) + d^*(A^* \cap B^*) + 1/n.$$

The other direction

$$d^*(A^*) + d^*(B^*) \geq d^*(A^*B^*) + d^*(A^* \cap B^*)$$

is clear by monotonicity.

So we have

$$d^*(A^*) + d^*(B^*) \approx d^*(A^*B^*) + d^*(A^* \cap B^*),$$

equivalently,

$$d^*(A^*/B^*) \approx d^*(A^*/A^* \cap B^*).$$

(2  $\rightarrow$  1): Fix any  $n \in \omega$ ,  $A_0 \subset_{\text{fin}} A$ , and  $B_0 \subset_{\text{fin}} B$ . Let  $A^* \supset_{\text{hf}} A$  and  $B^* \supset_{\text{hf}} B$  be a witness of  $d^*$ -independence. By the finite closure condition, we can take  $A^*$  and  $B^*$  to be both closed. Then  $A^*$  and  $B^*$  satisfy the following formula:

- $A_0 \subset \exists A^* \leq_{\text{fin}} A$ ,  $B_0 \subset \exists B^* \leq_{\text{fin}} B$ , and
- $d(A^*) + d(B^*) \leq d(A^*B^*) + d(A^* \cap B^*) + 1/n$ .

Because  $M$  is an elementary substructure of  $M^*$ , we can take expected sets.

(2  $\rightarrow$  3): Let  $A^*$  and  $B^*$  be witness of  $d^*$ -independence. Take any  $A' \subset_{\text{fin}} A$  and any  $B' \subset_{\text{fin}} B$ . Then  $d(A^*/B^*) \approx d(A^*/A^* \cap B^*)$ . By transposition,  $d(B^*/A^*) \approx d(B^*/A^* \cap B^*)$ . By monotonicity of  $d$ ,  $d(B^*/A'A^* \cap B^*) \approx d(B^*/A^* \cap B^*)$ . By transposition,  $d(A'/B^*) \approx d(A'/A^* \cap B^*)$ . By Monotonicity,  $d(A'/B'A^* \cap B^*) \approx d(A'/A^* \cap B^*)$ . By Lemma 9,  $d(A'/B'A \cap B) = d(A'/A \cap B)$ .

(3  $\rightarrow$  2): Take a closed hyperfinite extension  $A^*$  of  $A$  and a closed hyperfinite extension  $B^*$  of  $B$ . By compactness, it is enough to prove that for any  $A_0 \subset_{\text{fin}} A$ , the following set of formulas are satisfiable:

1.  $X \in F$
2.  $X \subseteq A$
3.  $A_0 \subseteq X$
4.  $X$  is closed
5.  $d(X/B^*) \approx d(X/X \cap B^*)$

We show  $A_0^* = \overline{A_0(A^* \cap B^*)}$  is a realization of the above set of formulas. 1, 2, 3, and 4 are clear.

5. First,

$$\begin{aligned} d(A_0^*/B^*) &= d(A_0^*B^*) - d(B^*) \\ &= d(A_0B^*) - d(B^*) \\ &= d(A_0/B^*) \\ &\approx d(A_0/B). \end{aligned}$$

Second,

$$\begin{aligned}
 d(A_0^*/A_0^* \cap B^*) &= d(A_0^*) - d(A_0^* \cap B^*) \\
 &= d(A_0(A^* \cap B^*)) - d(A_0^* \cap B^*) \\
 &\leq d(A_0(A^* \cap B^*)) - d(A^* \cap B^*) \\
 &= d(A_0/A^* \cap B^*) \\
 &\approx d(A_0/A \cap B)
 \end{aligned}$$

Finally, by the  $d$ -independence of  $A$  and  $B$ ,  $d(A_0/B) = d(A_0/A \cap B)$ .

Hence,  $d(A_0^*/A_0^* \cap B^*) \lesssim d(A_0^*/B^*)$ . The other direction is clear.

### Consequence

DS is equivalent to the first condition of DW. In particular, DW is a stronger condition than DS.

**Fact 13** [3] *Let  $T$  be stable. Then the following are equivalent:*

1.  $T$  is superstable.
2. For any  $B \subset \mathcal{M}$  and  $p \in S(B)$ , there is finite  $A \subseteq B$  such that  $p$  does not fork over  $A$ .

So, we have the following corollary.

**Corollary 14** *Suppose DS and that for any closed set  $A, B$ ,  $A \downarrow_{A \cap B}^d B$  if and only if  $A \downarrow_{A \cap B} B$ . Then  $T = \text{Th}(M)$  is  $\omega$ -stable or merely stable.*

This corollary is a partial solution of Baldwin's problem[1].

## References

- [1] J. T. Baldwin, Problems on pathological structures, In Helmut Wolter Martin Weese, editor, Proceedings of 10th Easter Conference in Model Theory (1993) 1-9
- [2] J. T. Baldwin and N. Shi, Stable generic structures, Annals of Pure and Applied Logic 79 (1996) 1-35
- [3] A. Pillay, An Introduction to Stability Theory, Oxford University Press, 1983
- [4] F. O. Wagner, Relational structures and dimensions, Kaye, Richard (ed.) et al., Automorphisms of first-order structures. Oxford: Clarendon Press. 153-180 (1994)